ISRAEL JOURNAL OF MATHEMATICS 83 (1993), 353-360

DIVISION ALGEBRAS WITH NO COMMON SUBFIELDS

BY

BILL JACOB*

Department of Mathematics, University of California at Santa Barbara
Santa Barbara, CA 93106, USA, email:jacob@math.ucsb.edu

AND

ADRIAN R. WADSWORTH*

Department of Mathematics, University of California at San Diego
9500 Gilman Drive, La Jolla, CA 92093-0112, USA, email:arwadsworth@ucsd.edu

ABSTRACT

An example is given of division algebras D_1 and D_2 of odd prime degree p over a field K such that D_1 and D_2 have no common subfield properly containing F, but $D_1^{\otimes i} \otimes_K D_2$ is not a division algebra for $1 \leq i \leq p-1$.

Let K be a field, and let D_1 and D_2 be division algebras with center K and finite-dimensional over K. If D_1 and D_2 each contain isomorphic copies of some field $L \supsetneqq K$, then $D_1 \otimes_K D_2$ is not a division algebra, since its subring $L \otimes_K L$ has zero divisors. If D_1 and D_2 are both quaternion algebras over K, then Albert [A] proved a converse: if $D_1 \otimes_K D_2$ is not a division algebra, then there is a common subfield of D_1 and D_2. After Albert's theorem was proved the question arose whether there is an analogous result for algebras of higher degree. There was no particular reason to expect a generalization of Albert's result, but the question was unsettled for many years. A negative answer was given in [TW, Prop. 5.1] where an example was constructed for each odd integer n of D_1 and D_2 each of degree n, such that $D_1 \otimes_K D_2$ was not a division algebra, but D_1 and D_2 had no common subfields. This was quickly deflated by D. Saltman, who pointed out that even though $D_1 \otimes_K D_2$ is not a division algebra in these

* Supported in part by the NSF.
Received March 11, 1992 and in revised form July 15, 1992

examples, $D_1 \otimes_K D_2^{\mathrm{op}}$ *is* a division algebra, where D_2^{op} is the opposite algebra of D_2. So, since D_2 and D_2^{op} have the same subfields, there can be no common subfields of D_1 and D_2. Saltman pointed out that a more reasonable question is: If $D_1^{\otimes i} \otimes_K D_2$ is not a division algebra for $1 \leq i < \exp(D_1)$, must D_1 and D_2 have a common subfield properly containing K? (Here $D_1^{\otimes i}$ is the underlying division algebra of $D_1 \otimes_K D_1 \otimes_K \cdots \otimes_K D_1$ (i times) and $\exp(D_1)$ is the order of the class of D_1 in the Brauer group of K.)

Counterexamples to this question have remained rather elusive. Mammone [M] recently produced a counterexample where D_1 has degree n and D_2 has degree n^2 for any integer $n > 1$. But the question is most tantalizing when D_1 and D_2 are required to have prime degree p. Note that in this case the hypothesis that $D_1^{\otimes i} \otimes_K D_2$ not be a division algebra for $1 \leq i < p$ is actually symmetric in D_1 and D_2 since it is equivalent to the Schur index condition: $\mathrm{ind}(D_1^{\otimes i} \otimes_K D_2^{\otimes j}) < p^2$ for all i, j. In this paper we settle this case by giving a counterexample for every odd prime p.

Our construction will make use of valuation theory on fields and on division algebras. We will use the following notation: if E is a field or a division algebra, Γ is an ordered abelian group, and $v\colon E^* \to \Gamma$ is a valuation, we will write V_E for the valuation ring of v on E, M_E for the unique maximal ideal of V_E, $\overline{E}$ for the residue division algebra V_E/M_E, and Γ_E for the value group $\mathrm{im}(v)$. If $a \in V_E$, we write $\overline{a}$ for the image of a in $\overline{E}$. We use the same letter v for an extension of v to a field or division algebra E' containing E—in the cases considered here, v will have only one extension to E'. All the commutative valuation theory we need is covered, e.g., in [E], while the noncommutative valuation theory needed can be found in [JW].

Now fix some prime number p. Our ground field F will always be assumed to contain a primitive p-th root of unity ω. For any $a, b \in F^* = F - \{0\}$, we write $(a, b\,; F)_p$ for the p^2-dimensional symbol algebra over F with generators i, j and relations, $i^p = a$, $j^p = b$, $ij = \omega ji$. (The choice of ω is fixed throughout.) For $a \in F^*$, we write $N_F(a)$ for the image of the norm map, $N_{F(\sqrt[p]{a})/F}(F(\sqrt[p]{a})^*)$. Our counterexample will consist of two division algebras over the Laurent series field $F((X))$.

Theorem 1: *Let $a, c, d \in F^*$ with $a, c \notin F^p$ and $F(\sqrt[p]{a}) \neq F(\sqrt[p]{c})$, and let $K = F((X))$. Let $D_1 = (a, X\,; K)_p$ and $D_2 = (c, dX\,; K)_p$. Then D_1 and D_2 are division algebras of degree p, and*

(i) *for* $1 \le i \le p-1$, $D_1^{\otimes i} \otimes_K D_2$ *is not a division algebra iff* $(c, d; F(\sqrt[p]{a^i c}))_p$ *is not a division algebra;*

(ii) D_1 *and* D_2 *have no common subfield strictly containing* K *iff* $d \notin N_F(a) \cdot N_F(c)$.

Proof: (i) The canonical discrete valuation v: $K^* \to \mathbb{Z}$ (with valuation ring $F[[X]]$) is complete, so Henselian; hence, this valuation extends uniquely to each division algebra finite-dimensional over K. Now, D_1 and D_2 are division algebras by [JW, Ex. 4.3] since $[F(\sqrt[p]{a})\colon F] = [F(\sqrt[p]{c})\colon F] = p$ and $v(X) = v(dX) = 1$, which has order p in $\Gamma_K/p\Gamma_K$. (This can also be seen by an easy norm calculation, or by viewing the D_i as twisted Laurent power series rings.) From the bilinearity of symbol algebras we have the Brauer group equivalences $D_1^{\otimes i} \otimes_K D_2 \sim (a^i, X; K)_p \otimes_K (c, dX; K)_p \sim N \otimes_K I$, where $N = (a^i c, X; K)_p$ and I is the underlying division algebra of $(c, d; K)_p$. Now, as $F(\sqrt[p]{a}) \neq F(\sqrt[p]{c})$, N is a division algebra for the same reason as D_1 and D_2, and with respect to the extension of v to N, N is "nicely semiramified" in the terminology of [JW, §4]; in particular, its residue division algebra $\overline{N}$ is the field $F(\sqrt[p]{a^i c})$. Also, since c and d are valuation units, I is inertial over K and $\overline{I}$ is the underlying division algebra of $(c, d; F)_p$ (cf. [JW, Ex. 2.4(i), Prop. 2.5]). By the Schur index formula in [JW, Th. 5.15(a)],

$$\operatorname{ind}(N \otimes_K I) = \operatorname{ind}(N) \cdot \operatorname{ind}(\overline{N} \otimes_{\overline{K}} \overline{I}) = p \cdot \operatorname{ind}(c, d; F(\sqrt[p]{a^i c}))_p.$$

Thus, $D_1^{\otimes i} \otimes_K D_2$ is not a division algebra iff $\operatorname{ind}(N \otimes_K I) < p^2$ iff $(c, d; F(\sqrt[p]{a^i c}))_p$ is split, proving (i).

(ii) Consider the subfields L of D_1 with $L \supsetneq K$. If L is unramified over K, then $[\overline{L}\colon \overline{F}] = [L\colon F] > 1$ and $\overline{L} \subseteq \overline{D_1} = F(\sqrt[p]{a})$. Hence, $\overline{L} = F(\sqrt[p]{a})$. On the other hand, if L is ramified over K, then $\Gamma_L = \Gamma_{D_1} = \frac{1}{p}\mathbb{Z}$, so $L = K(\sqrt[p]{uX})$ for some $u \in K^*$ with $v(u) = 0$. (Recall that a tame and totally ramified extension of a Henselian valued field is always a radical extension, cf. [S, p. 64, Th. 3].) If $\overline{u} = e \in \overline{K} = F$, then $e \equiv u \bmod K^{*p}$ (as $1 + M_K \subseteq K^{*p}$ since v is Henselian). So, $L = K(\sqrt[p]{eX})$. To see what possible e may occur, note that L splits D_1, so that $L \sim D_1 \otimes_K L \sim (a, X; L)_p \sim (a, e^{-1}; L)_p$, since $X \equiv e^{-1} \bmod L^{*p}$. Now, L is a totally ramified extension of K, so $\overline{L} = \overline{K} = F$. Since a, e are valuation units of L, the underlying division algebra of $(a, e^{-1}; L)_p$ is inertial with residue division algebra similar to $(a, e^{-1}; \overline{L})_p$. Thus, as L splits D_1, $(a, e^{-1}; F)_p$ must be split, so $e \in N_F(a)$. Conversely, if $e \in N_F(a)$, then $K(\sqrt[p]{eX})$ splits D_1 since $(a, X; K)_p \cong (a, eX; K)_p$. The subfields of D_2 are obtained by a similar calculation: An unramified subfield of D_2 has residue $F(\sqrt[p]{c})$. Also, since for $e \in F^*$, $D_2 \otimes_K K(\sqrt[p]{eX}) \cong (c, de^{-1}; K(\sqrt[p]{eX}))_p$, the totally ramified subfields of D_2 are those $K(\sqrt[p]{eX})$ with $(c, de^{-1}; F)_p$ split, i.e., $de^{-1} \in N_F(c)$. Thus, D_1 and D_2 have no common subfield unramified over K since we have assumed $F(\sqrt[p]{a}) \neq F(\sqrt[p]{c})$. But D_1 and D_2 have a common subfield ramified over K iff there is an $e \in N_F(a)$ with $de^{-1} \in N_F(c)$. This is equivalent to: $d \in N_F(a) \cdot N_F(c)$. ∎

Remarks: (1) If $p = 2$, then an easy calculation shows that if $(c, d;\, F(\sqrt{ac}))_2$ is split then $d \in N_F(a) \cdot N_F(c)$. So, Theorem 1 does not conflict with Albert's result. However, for any *odd* prime p, we will give an example of a field F and elements $a, c, d \in F$ satisfying the conditions in both (i) and (ii) of the Theorem. Thus, the corresponding D_1 and D_2 over $K = F((X))$ satisfy $D_1^{\otimes i} \otimes_K D_2$ is not a division algebra for $1 \leq i \leq p-1$ but D_1 and D_2 have no common subfield properly containing K.

(2) The conditions of the Theorem are preserved under prime-to-p extensions. That is, if we have F and $a, c, d \in F^*$ as specified in Theorem 1 such that the conditions in (i) and (ii) both hold and if E is any finite degree extension of K with $p \nmid [E\colon K]$, then for $D_i' = D_i \otimes_K E$, (a) D_1' and D_2' are division algebras of degree p, (b) each $D_1'^{\otimes i} \otimes_E D_2'$ is not a division algebra, and (c) D_1' and D_2' have no common subfields. Assertions (a) and (b) follow from the corresponding properties for D_1 and D_2, as $p \nmid [E\colon K]$. We obtain (c) by the same argument as for Th. 1 (ii), since the valuation v on K has a unique extension to a valuation on E, with $[\overline{E}\colon \overline{K}]\,|\Gamma_E\colon \Gamma_K| = [E\colon K]$. Since $p \nmid [\overline{E}\colon F]$ we find $\overline{E}(\sqrt[p]{a}) \neq \overline{E}(\sqrt[p]{c})$ and $d \notin N_{\overline{E}}(a) \cdot N_{\overline{E}}(c)$. Because $p \nmid |\Gamma_E\colon \Gamma_K|$, the image of X is a generator of $\Gamma_E / p\Gamma_E$. So, the argument goes through.

Example 2: Let p be any odd prime number and let F_0 be the completion of the rational function field $\mathbb{Q}(T)$ with respect to the discrete valuation which is the localization of $\mathbb{Z}[T]$ with respect to its minimal prime ideal $p\mathbb{Z}[T]$. So, there is a complete discrete valuation $v\colon F_0^* \to \mathbb{Z}$ with residue field $\overline{F_0} = \mathbb{F}_p(t)$ where t, the image of T, is transcendental over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and $v(p) = 1$. Let $F_1 = F_0(\omega)$, where ω is a primitive p-th root of unity. F_1 is a totally ramified extension of F_0 with $[F_1\colon F_0] = p-1$ (since $\mathbb{Q}(\omega)$ is a totally ramified extension of $\mathbb{Q}$ with respect to the p-adic valuation, cf. [W, p. 262]). Let F be any finite degree extension of F_1 such that F is totally ramified over F_1 with respect to v, $[F\colon F_1] \geq 3/(p-2)$, and $p \nmid [F\colon F_1]$. (So, for $p \geq 5$ we can take $F = F_1$, while for $p = 3$ we can take $F = F_1(\sqrt[4]{\pi_1})$, where π_1 is a uniformizing parameter for V_{F_1}.) For the extension of v to F we have

$$\overline{F} = \overline{F_0} = \mathbb{F}_p(t) \quad \text{and} \quad \Gamma_F = \frac{1}{k}\mathbb{Z},$$

where $k = [F\colon F_1](p-1) \geq 3(p-1)/(p-2)$, and $v(p) = 1 \notin p\Gamma_F$ since $p \nmid k$. Now choose $\theta \in F^*$ such that

$$\text{(1)} \qquad \frac{1}{p-1}\, v(p) < v(\theta) < v(p) \quad \text{and} \quad v(\theta) \notin p\Gamma_F\,.$$

For example, we could take $\theta = \pi^l$, where π is a uniformizing parameter for V_F and $k/(p-1) < l < k$ with $p \nmid l$. Such an l exists since

$$k - \frac{k}{p-1} \geq 3,$$

so there are at least two successive integers strictly between $k/(p-1)$ and k. Let

$$\text{(2)} \qquad a = T, \quad c = 1 + T, \quad \text{and} \quad d = 1 + \theta.$$

Since $t, 1+t \notin \mathbb{F}_p(t^p) = \overline{F}^p$, certainly $a, c \notin F^p$. Likewise, since $\overline{a^ic} = t^i(1+t) \notin \overline{F}^p$, we have $a^ic \notin F^p$ for all i, so $F(\sqrt[p]{a}) \neq F(\sqrt[p]{c})$ by Kummer theory. Lemmas 3 and 5 below will show that the a, c, d in (2) satisfy the conditions (i) and (ii) of Theorem 1.

LEMMA 3: *With F, a, c, d, as in Example 2, $(c, d; F(\sqrt[p]{a^ic}))_p$ is split for $1 \leq i \leq p-1$.*

Proof: We work for a moment in the subfield F_0 of F defined above, which has p for a uniformizing parameter. Let $\tau = \sqrt[p]{a^ic} = \sqrt[p]{T^i(1+T)}$ and let $E = F_0(\tau)$. With respect to the extension of v to E, τ is a valuation unit with residue $\overline{\tau} = \sqrt[p]{t^i(1+t)}$. So, E has no ramification over F_0, and

$$\overline{E} \supseteq \overline{F_0}(\overline{\tau}) = \mathbb{F}_p(t)(\sqrt[p]{t^i(1+t)}).$$

But since $\sqrt[p]{t^i(1+t)} \notin \mathbb{F}_p(t)$, $\mathbb{F}_p(t) \subsetneqq \mathbb{F}_p(t)(\sqrt[p]{t^i(1+t)}) \subseteq \mathbb{F}_p(\sqrt[p]{t})$. Since $[\mathbb{F}_p(\sqrt[p]{t})\colon \mathbb{F}_p(t)] = p$, we have $\mathbb{F}_p(\sqrt[p]{t}) = \mathbb{F}_p(t)(\sqrt[p]{t^i(1+t)}) \subseteq \overline{E}$. In particular, $\sqrt[p]{1+t} \in \overline{E}$. Thus, there is a valuation unit σ of E with $\overline{\sigma}^p = 1+t$. That is,

$$\sigma^p \equiv 1 + T \pmod{pV_E};$$

hence, $1+T=\sigma^p(1+pz)$ for some $z \in V_E$, where $\sigma \in E \subseteq F(\sqrt[p]{a^i c})$. Thus, the identities for symbol algebras [D, pp. 79–82] yield

(3)

$$\begin{aligned}(c,d;F(\sqrt[p]{a^ic}))_p &= (1+T,1+\theta;F(\sqrt[p]{a^ic}))_p\\ &= (\sigma^p(1+pz),1+\theta;F(\sqrt[p]{a^ic}))_p\\ &\sim (1+pz,1+\theta;F(\sqrt[p]{a^ic}))_p \otimes_{F(\sqrt[p]{a^ic})} (-\theta,1+\theta;F(\sqrt[p]{a^ic}))_p\\ &\sim (-\theta(1+pz),1+\theta;F(\sqrt[p]{a^ic}))_p\\ &\cong (1-\theta pz,(1+\theta)\theta^{-1}(1+pz)^{-1};F(\sqrt[p]{a^ic}))_p\ .\end{aligned}$$

(For the last step we used the identity $(r,s;L)_p \cong (r+s,-s/r;L)_p$, cf. [D, Lemma 11, p. 82].) However, $1+m \in F(\sqrt[p]{a^ic})^p$ whenever $v(m) > \frac{p}{p-1}v(p)$ as v on $F(\sqrt[p]{a^ic})$ is complete discrete, so Henselian. (For, if $v(m) > \frac{p}{p-1}v(p)$ write $m = u\cdot\pi^{v(m)/v(\pi)}$ with $v(u)=0$ and π a uniformizing parameter of $V_{F(\sqrt[p]{a^ic})}$; then

$$h(X) = \pi^{-v(m)/v(\pi)}[(1+\pi^{[v(m)-v(p)]/v(\pi)}X)^p - 1] - u \in V_{F(\sqrt[p]{a^ic})}[X]$$

has a root r in $V_{F(\sqrt[p]{a^ic})}$ by Hensel's Lemma since the image of h in $\overline{F(\sqrt[p]{a^ic})}[X]$ is $\overline{p\pi^{-v(p)/v(\pi)}}X-\overline{u}$. Then, $(1+\pi^{[v(m)-v(p)]/v(\pi)}r)^p = 1+m$.) Since we assumed in (1) that $v(\theta) > \frac{1}{p-1}v(p)$, we have $v(-\theta pz) > \frac{p}{p-1}v(p)$. Thus, $1-\theta pz \in F(\sqrt[p]{a^ic})^p$, so (3) shows that $(c,d;F(\sqrt[p]{a^ic}))_p$ is split. ∎

The following lemma is well-known, but we include a proof for lack of a convenient reference.

LEMMA 4: *Let $E=\mathbb{Q}(X_0,\dots,X_{p-1},Y)$ where $X_0,\dots,X_{p-1},Y$ are independent indeterminates over $\mathbb{Q}$ and p is prime, let ω be a primitive p-th root of unity, and let $L=E(Z)$ where $Z^p=Y$. Set*

$$f(X_0,\dots,X_{p-1},Z) = \prod_{i=0}^{p-1}\left(\sum_{j=0}^{p-1}X_j(\omega^iZ)^j\right) \in L(\omega)\ .$$

Then,

$$f(X_0,\dots,X_{p-1},Z) = X_0^p + X_1^pY + \dots + X_{p-1}^pY^{p-1} + p\,g(X_0,\dots,X_{p-1},Y)\ ,$$

where $g(X_0,\dots,X_{p-1},Y) \in \mathbb{Z}[X_0,\dots,X_{p-1},Y]$.

Proof: Let $\varphi = f(X_0,\dots,X_{p-1},Z)$; clearly φ is the norm $N_{L/E}\left(\sum_{j=0}^{p-1}X_jZ^j\right)$. So, as $\sum_{j=0}^{p-1}X_jZ^j$ is integral over $\mathbb{Z}[X_0,\dots,X_{p-1},Y]$ which is integrally closed,

$\varphi \in \mathbb{Z}[X_0, \dots, X_{p-1}, Y]$. Let $R = \mathbb{Z}[\omega]$, which is integral over $\mathbb{Z}$, and let M be a maximal ideal of R with $M \cap \mathbb{Z} = p\mathbb{Z}$. Since the image of ω in R/M equals 1 as $char(R/M) = p$, the image of φ in $R/M[X_0, \dots, X_{p-1}, Z]$ is $\left(\sum_{j=0}^{p-1} X_j Z^j\right)^p = \sum_{j=0}^{p-1} X_j^p Y^j$. Hence,

$$\begin{aligned}\varphi - \sum_{j=0}^{p-1} X_j^p Y^j &\in \ker(\mathbb{Z}[X_0, \dots, X_{p-1}, Y] \to R/M[X_0, \dots, X_{p-1}, Z]) \\ &= p\mathbb{Z}[X_0, \dots, X_{p-1}, Y],\end{aligned}$$

as desired. ∎

LEMMA 5: *With F, a, c, d as in Example 2, $d \notin N_F(a) \cdot N_F(c)$.*

Proof: Let $L_1 = F(\sqrt[p]{T})$ and $L_2 = F(\sqrt[p]{1+T})$; since $t, 1+t \notin \mathbb{F}_p(t)^p$, the unique extension of v to each L_i has ramification index 1 with $\overline{L_1} = \overline{F}(\sqrt[p]{t}) = \overline{F}(\sqrt[p]{1+t}) = \overline{L_2}$. Since the valuation extends uniquely, we have $v(N_{L_i/F}(\gamma)) = pv(\gamma)$ for all $\gamma \in L_i^*$. Let $V_i = V_{L_i}$, the valuation ring of v on L_i, and let $\tau = \sqrt[p]{T} \in V_1$. Because $[\overline{L_1}\colon \overline{F}] = [L_1\colon F]$ with $\overline{L_1} = \overline{F}(\overline{\tau})$, V_1 is a free V_F-module with base $\{1, \tau, \tau^2, \dots, \tau^{p-1}\}$ (cf. [E, Th. 18.6]). For any $\alpha \in V_1$, write $\alpha = \sum_{j=0}^{p-1} s_j \tau^j$ with $s_j \in V_F$. Then, with f the norm polynomial of Lemma 4, we have

$$\begin{aligned}N_{L_1/F}(\alpha) &= f(s_0, \dots, s_{p-1}, \tau) \\ &= s_0^p + s_1^p T + \dots + s_{p-1}^p T^{p-1} + p\,g(s_0, \dots, s_{p-1}, T) \\ &\equiv s_0^p + s_1^p T + \dots + s_{p-1}^p T^{p-1} \pmod{pV_F}\,.\end{aligned} \tag{4}$$

Since $(s_i + s_i')^p \equiv s_i^p + s_i'^p \pmod{pV_F}$, it follows from (4) that for any $\alpha, \alpha' \in V_1$,

$$N_{L_1/F}(\alpha + \alpha') \equiv N_{L_1/F}(\alpha) + N_{L_1/F}(\alpha') \pmod{pV_F}\,. \tag{5}$$

Because the norm is always multiplicative, (5) shows that $N_{L_1/F}$ induces a ring homomorphism $V_1 \to V_F/pV_F$; let R be the image of this map in V_F/pV_F.

Turning now to V_2, and setting $\sigma = (1+T)^{1/p}$, we have likewise that V_2 is a free V_F-module with base $\{1, \sigma, \dots, \sigma^{p-1}\}$. Lemma 4 shows that for $\beta = \sum_{j=0}^{p-1} r_j \sigma^j \in V_2$ (so each $r_j \in V_F$),

$$N_{L_2/F}(\beta) \equiv r_0^p + r_1^p(1+T) + \cdots + r_{p-1}^p(1+T)^{p-1} \pmod{pV_F}\,. \tag{6}$$

By expanding out the powers of $1+T$ in (6) and invoking the additivity of p-th powers mod p, we see from (4) that $N_{L_2/F}(V_2)$ maps into R in V_F/pV_F.

Now, suppose $1+\theta = N_{L_1/F}(\alpha)\cdot N_{L_2/F}(\beta)$ for some $\alpha \in L_1^*$, $\beta \in L_2^*$. Then, $0 = v(1+\theta) = pv(\alpha)+pv(\beta)$, so $v(\beta) = -v(\alpha)$. Thus, after replacing α by $r^{-1}\alpha$ and β by $r\beta$ where $r \in F^*$ with $v(r) = v(\alpha) \in \Gamma_{L_1} = \Gamma_F$, we may assume $v(\alpha) = v(\beta) = 0$. So, $1+\theta$ maps into the subring R of V_F/pV_F; hence θ also maps into R. That is, there is $\gamma \in V_1$ and $e \in V_F$ with $N_{L_1/F}(\gamma) = \theta + pe$. But this cannot occur since $v(N_{L_1/F}(\gamma)) = pv(\gamma) \in p\Gamma_{L_1} = p\Gamma_F$, while since $v(\theta) < v(p)$ by (1), $v(\theta+pe) = v(\theta) \notin p\Gamma_F$, again by (1). This contradiction shows $d = 1+\theta \notin N_F(a)\cdot N_F(c)$, as desired. ∎

References

[A] A. A. Albert, *On the Wedderburn norm condition for cyclic algebras*, Bull. Amer. Math. Soc. **37** (1931), 301–312.

[D] P. Draxl, *Skew Fields*, London Math. Soc. Lecture Note Series, No. 81, Cambridge Univ. Press, Cambridge, England, 1983.

[E] O. Endler, *Valuation Theory*, Springer-Verlag, New York, 1972.

[JW] B. Jacob and A. R. Wadsworth, *Division algebras over Henselian fields*, J. Algebra **125** (1990), 126–179.

[M] P. Mammone, *On the tensor product of division algebras*, Arch. Math. **58** (1992), 34–39.

[S] O. F. G. Schilling, *The Theory of Valuations*, Math. Surveys, No. 4, Amer. Math. Soc., Providence, R.I., 1950.

[TW] J.-P. Tignol and A. R. Wadsworth, *Totally ramified valuations on finite-dimensional division algebras*, Trans. Amer. Math. Soc. **302** (1987), 223–250.

[W] E. Weiss, *Algebraic Number Theory*, McGraw-Hill, New York, 1963.